Lecture 18

Distinction of "s" and "5" in Lecture notes

"s" $\downarrow$ letter

"5" $\downarrow$ five

- Most times it should be "s" not "5".
- "5" seldom appears (possibly as coefficients of some P.E) in the Lectures

Lecture 18

Plan: §7.7/7.8 Convolutions

Q: If we know $\begin{cases} F = \mathcal{L}\{f\} \ (\Rightarrow f = \mathcal{L}^{-1}\{F\}) \\ G = \mathcal{L}\{g\} \ (\Rightarrow g = \mathcal{L}^{-1}\{G\}) \end{cases}$

Can we compute $\mathcal{L}^{-1}\{F \cdot G\}$.

A: Yes! And we will need to use "Convolutions".

Now what is "convolution"?

<u>Def^n</u>: Let $f(t)$, $g(t)$ be piecewise continuous functions on $[0, \infty)$. Then convolution of $f(t)$ and $g(t)$ is: $\quad$ denoted by $f * g$

$$(f*g)(t) = \int_0^t f(t-x)\, g(x)\, dx.$$

Proposition: (properties of Convolution)

Let $f(t)$, $g(t)$, $h(t)$ be piecewise continuous functions on $[0,\infty)$. Then

① $f*g = g*f$

$k_1, k_2 \in \mathbb{R}$

② $f*(k_1 g + k_2 h) = k_1(f*g) + k_2(f*h)$

③ $(f*g)*h = f*(g*h)$

Pf of ①:

$$LHS = f*g(t)$$

$$= \int_0^t f(t-x)\,g(x)\,dx$$

use u-sub: $u = t-x$ $\quad$ t: fixed

$\Rightarrow$ $x = t-u$; $du = -dx$

when $x=0$, $u=t-0=t$; when $x=t$, $u=t-t=0$

$$\Rightarrow \quad LHS = \int_t^0 f(u)\, g(t-u)\,(-du)$$

$$= \int_0^t f(u) g(t-u)\, du$$

$$RHS = g * f(t)$$

$$= \int_0^t g(t-x) f(x)\, dx$$

$$= \int_0^t f(x)\, g(t-x)\, dx$$

$$\Rightarrow \quad LHS = RHS.$$

The main theorem of this section is:

Thm: Let $f(t), g(t)$ be piecewise continuous functions on $[0, \infty)$ with exponential order α and write $F(s) = \mathcal{L}\{f\}(s)$, $G(s) = \mathcal{L}\{g\}(s)$.

for $s > \alpha$

Then

$$\mathcal{L}\{f*g\}(s) = F(s)\cdot G(s) \quad (1)$$

or equivalent:

$$\mathcal{L}^{-1}\{F(s)\,G(s)\}(t) = (f*g)(t)$$

Pf: To prove (1), we note

$$f*g(t) = \int_0^t f(t-x)\,g(x)\,dx$$

$\Rightarrow$ (Recall $\mathcal{L}\{\underline{h}(t)\}(s) = \int_0^\infty e^{-st}\,h(t)\,dt$)

$$\mathcal{L}\{\underbrace{f*g}_{h}\}(s) = \int_0^\infty e^{-st}\Big(\underbrace{\int_0^t f(t-x)\,g(x)\,dx}_{f*g}\Big)dt$$

$$= \int_0^\infty e^{-st}\Big(\int_0^\infty u(t-x)\,f(t-x)\,g(x)\,dx\Big)\,dt$$

Why? This is because $u(t-x) = \begin{cases} 0, & t-x<0 \\ 1, & t-x\ge 0 \end{cases}$

$$= \begin{cases} 0, & x>t \\ 1, & x\le t \end{cases}$$

Recall

$$u(y) = \begin{cases} 0, & \text{if } y<0 \\ 1, & \text{if } y\ge 0 \end{cases}$$

Use $\int_0^\infty = \int_0^t + \cancel{\int_t^\infty}$

$$= \int_0^\infty g(x)\left(\int_0^\infty e^{-st} u(t-x) f(t-x)\,dt\right) dx$$

$\int_0^\infty = \cancel{\int_0^x} + \int_x^\infty$

why? we interchange the order of integration.

$$= \int_0^\infty g(x)\left(\int_x^\infty e^{-st} f(t-x)\,dt\right) dx$$

why? we used $u(t-x) = \begin{cases} 0, & t < x \\ 1, & t \geq x \end{cases}$

$$= \int_0^\infty g(x)\left(\int_0^\infty e^{-s(u+x)} f(u)\,du\right) dx$$

why? use u-sub: $u = t - x$

$$\Rightarrow t = u + x, \quad du = dt$$

$$= \int_0^\infty g(x) \left(\int_0^\infty e^{-su} e^{-sx} f(u)\, du \right) dx$$

$$= \int_0^\infty g(x) e^{-sx} \left(\underbrace{\int_0^\infty e^{-su} f(u)\, du}_{\mathcal{L}\{f\}(s)} \right) dx$$

$$= \int_0^\infty g(x)\, e^{-sx}\, F(s)\, dx$$

$$= F(s) \int_0^\infty g(x) e^{-sx}\, dx$$

$$= F(s)\, \mathcal{L}\{g\}(s) = F(s)\, G(s) \qquad \text{Q.E.D.}$$

Remark: we can remember the Theorem as:

① $\mathcal{L}\{f * g\} = \underbrace{\mathcal{L}\{f\}}_{F} \cdot \underbrace{\mathcal{L}\{g\}}_{G}.$

② $\mathcal{L}^{-1}\{F \cdot G\} = \left(\underbrace{\mathcal{L}^{-1}\{F\}}_{f}\right) * \left(\underbrace{\mathcal{L}^{-1}\{G\}}_{g}\right)$

Remark: The above pf is NOT required.

E.g 1: prove $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{\sin t - t\cos t}{2}$

we gave this as Hint in #HW5, Q5.

Pf: Note $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2+1}}_{F} \cdot \underbrace{\frac{1}{s^2+1}}_{G}\right\}$

$= \left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right) * \left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right)$

$= \sin t * \sin t$

$= \int_0^t \underbrace{\sin(t-x)\sin x}\, dx$

How to compute?

Recall from the table:

$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$, $b \in \mathbb{R}$

Hint:

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\Rightarrow \cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta$$

Let $\alpha = t-x,\ \beta = x$

$$\Rightarrow \cos(t-2x)-\cos(t)=2\sin(t-x)\sin x$$

$$\Rightarrow \sin(t-x)\sin x = \frac{1}{2}\left(\cos(t-2x)-\cos t\right)$$

Hence

$$\int_0^t \sin(t-x)\sin x\,dx$$

$$= \frac{1}{2}\int_0^t \left(\cos(t-2x)-\cos t\right)dx$$

$$= \frac{1}{2}\left(\int_0^t \cos(t-2x)\,dx - \int_0^t \cos t\,dx\right)$$

$$= \frac{1}{2}\left(\int_0^t \cos(2x-t)\,dx - \underbrace{\int_0^t \cos t\,dx}\right)$$

In the "eyes" of dx, $\cos t$ is constant!

$$= \frac{1}{2}\left(\left[\frac{\sin(2x-t)}{2}\right]\Big|_0^t - (\cos t)\cdot t\right)$$

E.X

$$= \frac{\sin t - t\cos t}{2}$$

Hence $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{\sin t - t\cos t}{2}$

E.g 2: Solve the following I.V.P:

$$\begin{cases} y'(t) = 1 - \underbrace{\int_0^t y(t-v)\, e^{-2v}\, dv}_{y(t) * e^{-2t}} . & (2) \\ y(0) = 1 \end{cases}$$

$$" \int_0^t y(t-x) \cdot e^{-2x}\, dx$$

Remark: Such a D.E in (2) is called an "integro-differential eqn" means a D.E that involves an integral.

A: First note the eqn (2) can be re-written as

$$y'(t) = 1 - y(t) * e^{-2t} \qquad (3)$$

Apply Laplace transform $\mathcal{L}$ to (3)

$\Rightarrow \quad \underset{\substack{\shortparallel \\ s\mathcal{L}\{y\}-y(0)}}{\mathcal{L}\{y'\}} = \mathcal{L}\{1\} - \mathcal{L}\{y(t) * e^{-2t}\}$

Recall Thm 2 §7.3

$\mathcal{L}\{f'\}(s)$

$= s\mathcal{L}\{f\}(s) - f(0)$

$\Rightarrow$

$\mathcal{L}\{y'\}(s)$

$= s\mathcal{L}\{y\}(s) - y(0)$

Table §7.2

$\mathcal{L}\{c\} = \frac{c}{s}$

$\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$

Write $Y = \mathcal{L}\{y\} \Rightarrow$

$$sY - 1 = \frac{1}{s} - \mathcal{L}\{y\} \cdot \mathcal{L}\{e^{-2t}\}$$

$$= \frac{1}{s} - Y \cdot \frac{1}{s+2}$$

$\Rightarrow$

$$\left(s + \frac{1}{s+2}\right) Y = \frac{1}{s} + 1$$

$\Rightarrow \quad \frac{(s+1)^2}{(s+2)} Y = \frac{1+s}{s}$

$\Rightarrow$

$$\frac{s+1}{s+2} Y = \frac{1}{s}$$

$\Rightarrow$

$$Y = \frac{s+2}{s(s+1)}$$

Then find the partial fractional decomposition of $\frac{s+2}{s(s+1)}$:

$$Y = \frac{s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$\cdots \Rightarrow \begin{cases} A = 2 \\ B = -1 \end{cases}$$

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Hence

$$Y = \frac{2}{s} - \frac{1}{s+1}$$

$$\Rightarrow y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{1}{s+1}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 2 \cdot 1 - e^{-t} = 2 - e^{-t}$$

Here we used the table in §7.2

$\mathcal{L}\{1\} = \frac{1}{s}$; $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$.